

Finite groups with permutable Hall subgroups

Xia Yin, Nanying Yang*

School of Science, Jiangnan University

Wuxi 214122 , P. R. China

E-mail:yangny@jiangnan.edu.cn

Abstract

Let $\sigma = \{\sigma_i | i \in I\}$ be a partition of the set of all primes \mathbb{P} and G a finite group. A set \mathcal{H} of subgroups of G is said to be a *complete Hall σ -set* of G if every member $\neq 1$ of \mathcal{H} is a Hall σ_i -subgroup of G for some $i \in I$ and \mathcal{H} contains exactly one Hall σ_i -subgroup of G for every i such that $\sigma_i \cap \pi(G) \neq \emptyset$.

In this paper, we study the structure of G assuming that some subgroups of G permutes with all members of \mathcal{H} .

1 Introduction

Throughout this paper, all groups are finite and G always denotes a finite group. We use $\pi(G)$ to denote the set of all primes dividing $|G|$. A subgroup A of G is said to *permute* with a subgroup B if $AB = BA$. In this case they say also that the subgroups A and B are *permutable*.

Following [1], we use σ to denote some partition of \mathbb{P} . Thus $\sigma = \{\sigma_i | i \in I\}$, where $\mathbb{P} = \cup_{i \in I} \sigma_i$ and $\sigma_i \cap \sigma_j = \emptyset$ for all $i \neq j$.

A set \mathcal{H} of subgroups of G is a *complete Hall σ -set* of G [2, 3] if every member $\neq 1$ of \mathcal{H} is a Hall σ_i -subgroup of G for some $\sigma_i \in \sigma$ and \mathcal{H} contains exactly one Hall σ_i -subgroup of G for every i such that $\sigma_i \cap \pi(G) \neq \emptyset$. If every two members of \mathcal{H} are permutable, then \mathcal{H} is said to be a *σ -basis* [4] of G . In the case when $\mathcal{H} = \{\{2\}, \{3\}, \dots\}$ a complete Hall σ -set \mathcal{H} of G is also called a *complete set of Sylow subgroups* of G .

We use \mathfrak{H}_σ to denote the class of all soluble groups G such that every complete Hall σ -set of G forms a σ -basis of G .

*Research is supported by a NNSF grant of China (Grant #11301227) and Natural Science Foundation of Jiangsu Province (grant # BK20130119).

⁰Keywords: finite group, Hall subgroup, complete Hall σ -set, permutable subgroups, supersoluble group.

⁰Mathematics Subject Classification (2010): 20D10, 20D15

A large number of publications are connected with study the situation when some subgroups of G permute with all members of some fixed complete set of Sylow subgroups of G . For example, the classical Hall's result states: *G is soluble if and only if it has a Sylow basis, that is, a complete set of pairwise permutable Sylow subgroups.* In [5] (see also Paragraph 3 in [6, VI]), Huppert proved that G is a soluble group in which every complete set of Sylow subgroups forms a Sylow basis if and only if the automorphism group induced by G on every its chief factor H/K has the order divisible by at most one different from p prime, where $p \in \pi(H/K)$. In the paper [7], Huppert proved that if G is soluble and it has a complete set \mathcal{S} of Sylow subgroups such that every maximal subgroup of every subgroup in \mathcal{S} permutes with all other members of \mathcal{S} , then G is supersoluble.

The above-mentioned results in [5, 6, 7] and many other related results make natural to ask:

(I) *Suppose that G has a complete Hall σ -set \mathcal{H} such that every maximal subgroup of any subgroup in \mathcal{H} permutes with all other members of \mathcal{H} . What we can say then about the structure of G ? In particular, does it true then that G is supersoluble in the case when every member of \mathcal{H} is supersoluble?*

(II) *Suppose that G possesses a a complete Hall σ -set. What we can say then about the structure of G provided every complete Hall σ -set of G forms a σ -basis in G ?*

Our first observation is the following result concerning Question (I).

Theorem A. *Suppose that G possesses a a complete Hall σ -set \mathcal{H} all whose members are supersoluble. If every maximal subgroup of any non-cyclic subgroup in \mathcal{H} permutes with all other members of \mathcal{H} , then G is supersoluble.*

In the classical case, when $\sigma = \{\{2\}, \{3\}, \dots\}$, we get from Theorem A the following two known results.

Corollary 1.1 (Asaad M., Heliel [8]). *If G has a complete set \mathcal{S} of Sylow subgroups such that every maximal subgroup of every subgroup in \mathcal{S} permutes with all other members of \mathcal{S} , then G is supersoluble.*

Note that Corollary 1.1 is proved in [8] on the base of the classification of all simple non-abelian groups. The proof of Theorem A does not use such a classification.

Corollary 1.2 (Huppert [6, VI, Theorem 10.3]). *If every Sylow subgroup of G is cyclic, then G is supersoluble.*

The class $1 \in \mathfrak{F}$ of groups is said to be a *formation* provided every homomorphic image of $G/G^{\mathfrak{F}}$ belongs to \mathfrak{F} . The formation \mathfrak{F} is said to be: *saturated* provided $G \in \mathfrak{F}$ whenever $G^{\mathfrak{F}} \leq \Phi(G)$; *hereditary* provided $G \in \mathfrak{F}$ whenever $G \leq A \in \mathfrak{F}$.

Now let $p > q > r$ be primes such that qr divides $p - 1$. Let P be a group of order p and $QR \leq \text{Aut}(P)$, where Q and R are groups with order q and r , respectively. Let $G = P \rtimes (QR)$. Then, in view of the above-mentioned Hupper's result in [5], G is not a group such that every complete set of Sylow subgroups forms a Sylow basis of G . But it is easy to see that every complete

Hall σ -set of G , where $\sigma = \{\{2, 3\}, \{7\}, \{2, 3, 7\}'\}$, is a σ -basis of G . This elementary example is a motivation for our next result, which gives the answer to Question (II) in the universe of all soluble groups.

Theorem B. *The class \mathfrak{H}_σ is a hereditary formation and it is saturated if and only if $|\sigma| \leq 2$. Moreover, $G \in \mathfrak{H}_\sigma$ if and only if G is soluble and the automorphism group induced by G on every its chief factor of order divisible by p is either a σ_i -group, where $p \notin \sigma_I$, or a $(\sigma_i \cup \sigma_j)$ -group for some different σ_i and σ_j such that $p \in \sigma_i$.*

In the case when $\sigma = \{\{2\}, \{3\}, \dots\}$ we get from Theorem B the following

Corollary 1.3 (Huppert [5]). *Every complete set of Sylow subgroups of a soluble group G forms a Sylow basis of G if and only if the automorphism group induced by G on every its chief factor H/K has order divisible by at most one different from p prime, where $p \in \pi(H/K)$.*

2 Proof of Theorem A

Lemma 2.1 (See Knyagina and Monakhov [12]). *Let H , K and N be pairwise permutable subgroups of G and H is a Hall subgroup of G . Then $N \cap HK = (N \cap H)(N \cap K)$.*

Proof of Theorem A. Assume that this theorem is false and let G be a counterexample of minimal order. Let $\mathcal{H} = \{H_1, \dots, H_t\}$. We can assume, without loss of generality, that the smallest prime divisor p of $|G|$ belongs to $\pi(H_1)$. Let P be a Sylow p -subgroup of H_1 .

(1) *If R is a minimal normal subgroup of G , then G/R is supersoluble. Hence R is the unique minimal normal subgroup of G , R is not cyclic and $R \not\leq \Phi(G)$.*

We show that the hypothesis holds for G/R . First note that

$$\mathcal{H}_0 = \{H_1R/R, \dots, H_tR/R\}$$

is a complete Hall σ -set of G/R , where $H_iR/R \simeq H_i/H_i \cap R$ is supersoluble since H_i is supersoluble by hypothesis for all $i = 1, \dots, t$.

Now let V/R be a maximal subgroup of H_iR/R , so $|(H_iR/R) : (V/R)| = p$ is a prime. Then $V = R(V \cap H_i)$ and hence

$$\begin{aligned} p &= |(H_iR/R) : (V/R)| = |(H_iR/R) : (R(V \cap H_i)/R)| = |H_iR : R(V \cap H_i)| = \\ &= |H_i||R||R \cap (V \cap H_i)| : |V \cap H_i||R||H_i \cap R| = |H_i| : |V \cap H_i| = |H_i : (V \cap H_i)|, \end{aligned}$$

so $V \cap H_i$ is a maximal subgroup of H_i . Assume that H_iR/R is not cyclic. Then H_i is not cyclic, so

$$(V \cap H_i)H_j = H_j(V \cap H_i)$$

for all $j \neq i$ by hypothesis and hence

$$(V/R)(H_jR/R) = (R(V \cap H_i)/R)(H_jR/R) = (H_jR/R)((V \cap H_i)R/R) = (H_jR/R)(V/R).$$

Consequently the hypothesis holds for G/R , so G/R is supersoluble by the choice of G . Moreover, it is well known that the class of all supersoluble groups is a saturated formation (see Ch. VI in [6] or ??? in [?]). Hence the choice of G implies that R is the unique minimal normal subgroup of G , R is not cyclic and $R \not\leq \Phi(G)$.

(2) G is not soluble. Hence R is not abelian and $2 \in \pi(R)$.

Assume that this is false. Then R is an abelian q -group for some prime q . Let $q \in \pi_k$. Since R is non-cyclic by Claim (1) and $R \leq H_k$, H_k is non-cyclic. Hence every member of \mathcal{H} permutes with each maximal subgroup of H_k . Since $R \not\leq \Phi(G)$, $R \not\leq \Phi(H_k)$ and so there exists a maximal subgroup V of H_k such that $R \not\leq V$ and $RV = H_k$. Hence $E = R \cap V \neq 1$ since $|R| > q$ and H_k is supersoluble. Clearly, E is normal in H_k . Now assume that $i \neq k$. Then V permutes with H_i by hypothesis, so VH_i is a subgroup of G and

$$R \cap VH_i = (R \cap V)(R \cap H_i) = R \cap V = E$$

by Lemma 2.1 and so $H_i \leq N_G(E)$. Therefore $H_i \leq N_G(E)$ for all $i = 1, \dots, t$. This implies that E is normal in G , which contradicts the minimality of R . Hence we have (2).

(3) If R has a Hall $\{2, q\}$ -subgroup for each q dividing $|R|$, then a Sylow 2-subgroup R_2 of R is non-abelian.

Assume that this is false. Then by Claim (2) and Theorem 13.7 in [9, XI], the composition factors of R are isomorphic to one of the following groups: a) $PSL(2, 2^f)$; b) $PSL(2, q)$, where 8 divides $q - 3$ or $q - 5$; c) The Janko group J_1 ; d) A Ree group. But with respect to each of these groups it is well-known (see, for example [10, Theorem 1]) that the group has no a Hall $\{2, q\}$ -subgroup for at least one odd prime q dividing its order. Hence we have (3).

(4) If at least one of the subgroups H_i or H_k , say H_i , is non-cyclic, then $H_i H_k = H_k H_i$ (This follows from the fact that every maximal subgroup of H_i permutes with H_k).

(5) $H = H_1$ is not cyclic (This directly follows from Claim (2), [6, IV, 2.8] and the Feit-Thompson theorem).

In view of Claim (5), \mathcal{H} contains non-cyclic subgroups. Without loss of generality, we may assume that H_1, \dots, H_r are non-cyclic groups and all groups H_{r+1}, \dots, H_t are cyclic.

(6) Let $E_{\{i,j\}} = H_i H_j$ where $i \leq r$. If r is the smallest prime dividing $|E_{\{i,j\}}|$, then $E_{\{i,j\}}$ is p -nilpotent, so it is soluble. Therefore $E_{\{i,j\}} \neq G$.

Clearly, the hypothesis holds for $E_{\{i,j\}}$. Hence if $E_{\{i,j\}} < G$, then this subgroup is supersoluble by the choice of G , and so it is p -nilpotent. Now assume that $E_{\{i,j\}} = G$. Then $r = p = 2$ and $E_{\{i,j\}} = HH_j = H_j H$. Let V_1, \dots, V_t be the set of all maximal subgroups of a Sylow 2-subgroup P of H . Since H is supersoluble, it has a normal 2-complement S . Then SV_i is a maximal subgroup of H , so $SV_i H_j = H_j SV_i$ is a subgroup of G by hypothesis. Moreover, this subgroup is normal in $G = E_{\{i,j\}}$ since $|G : H_j SV_i| = 2$. Now let $E = SV_1 H_j \cap \dots \cap SV_t H_j$. Then E is normal in G and clearly $E \cap P \leq \Phi(P)$.

Now we show that for any prime q dividing $|H_j|$, there are a Sylow q -subgroup Q of H_j and an element $h \in H$ such that $P \leq N_G(Q^h)$. Indeed, by the Frattini argument, $G = EN_G(Q)$. Hence by [6, VI, 4.7], there are Sylow 2-subgroups G_2 , E_2 and N_2 of G , E and $N_G(Q)$ respectively such that $G_2 = E_2N_2$. Let $P = (G_2)^x$. Then $P = (E_2)^x(N_2)^x$, where $(E_2)^x$ is a Sylow 2-subgroup of E and $(E_2)^x$ is a Sylow 2-subgroup of $(N_G(Q))^x = N_G(Q^x)$. Since $G = HH_j$, $x = hw$ for some $h \in H$ and $w \in H_j$. Hence

$$N_G(Q^x) = N_G(Q^{wh}) = N_G((Q^w)^h),$$

where Q^w is a Sylow q -subgroup of H_j . Therefore $(E_2)^x = E \cap P \leq \Phi(P)$. Consequently, $P \leq N_G((Q^w)^h)$. This shows that for any prime q dividing $|H_j|$, there is a Sylow q -subgroup Q of H_j and an element $h \in H$ such that $P \leq N_G(Q^h)$. Thus G has a Hall $\{2, q\}$ -subgroup PQ^h for each q dividing $|H_j|$. Moreover, since H is supersoluble by hypothesis, G has a Hall $\{2, s\}$ -subgroup for each s dividing $|H|$. Hence in view of Claim (3), P is not abelian. Then $P \cap F(H) \neq 1$, so $P \cap F(H) \leq Z_\infty(H)$ since H is supersoluble. Let Z be a group of order 2 in $Z(H)$. Since $Z \leq P \leq N_G((Q^h)$, $Z = Z^{h^{-1}} \leq N_G(Q)$. It follows that $Z \leq N_G(H_j)$. Thus $Z^G = Z^{HH_j} = Z^{H_j} \leq ZH_j$. This shows that a Sylow 2-subgroup of Z^G has order 2. Hence Z^G is 2-nilpotent. Let S be the 2-complement of Z^G . It is clear that $S \neq 1$. Since S is characteristic in Z^G , it is normal in G . On the other hand, S is soluble by the Feit-Thompson theorem. This induces that G has an abelian minimal normal subgroup, which contradicts Claim (2). Thus (6) holds.

(7) $E_i = HH_i$ is supersoluble for all $i = 2, \dots, t$ ((Since the hypothesis holds for E_i and $E_i < G$ by Claim (5), this follows from the choice of G).

(8) $E = H_1 \dots H_r$ is soluble.

We argue by induction on r . If $r = 2$, it is true by Claim (5). Now let $r > 2$ and assume that the assertion is true for $r - 1$. Then by Claim (4), E has at least three soluble subgroups E_1, E_2, E_3 whose indices $|E : E_1|, |E : E_2|, |E : E_3|$ are pairwise coprime. But then E is soluble by the Wielandt theorem [11, I, 3.4].

(9) R has a Hall $\{2, q\}$ -subgroup for each q dividing $|R|$.

It is clear in the case when $q \in \pi(H)$. Now assume that $q \in \pi(H_i)$ for some $i > 1$. Then Claim (6) implies that $B = HH_i$ is a Hall soluble subgroup of G . Hence B has a Hall $\{2, q\}$ -subgroup V and so $V \cap R$ is a Hall $\{2, q\}$ -subgroup of R .

(10) A Sylow 2-subgroup R_2 of R is non-abelian (This follows from Claims (3) and (9)).

(11) If $q \in \pi(H_k)$ for some $k > r$, then q does not divide $|R : N_R((R_2)')|$.

By Claim (7), $B = HH_k$ is supersoluble. Hence there is a Sylow q -subgroup Q of B such that PQ is a Hall $\{2, q\}$ -subgroup of B . Then $U = PQ \cap R = (P \cap R)(Q \cap R) = R_2(Q \cap R)$ is a Hall supersoluble subgroup of R with cyclic Sylow q -subgroup $Q \cap R$. By [6, VI, 9.1], $Q \cap R$ is normal in U , and $U/C_U(Q \cap R)$ is an abelian group by [13, Ch. 5, 4.1]. Hence

$$R_2C_U(Q \cap R)/C_U(Q \cap R) \simeq R_2/R_2 \cap C_U(Q \cap R)$$

is abelian and so $(R_2)' \leq C_U(Q \cap R)$. Consequently, $Q \cap R \leq N_R((R_2)')$.

The final contradiction. In view of Claim (11), $R = (E \cap R)N_R((R_2)').$ Hence

$$((R_2)')^R = ((R_2)')^{(E \cap R)N_R((R_2)')} = ((R_2)')^{E \cap R} \leq E \cap R.$$

But by Claim (8), $E \cap R$ is soluble. On the other hand, Claim (10) implies that $(R_2)' \neq 1$ and so R is soluble, contrary to Claim (2). The theorem is thus proved.

3 Proof of Theorem B

The following lemma can be proved by the direct calculations on the base of well-known properties of Hall subgroups of soluble subgroups.

Lemma 3.1. *The class \mathfrak{H}_σ is closed under taking homomorphic images, subgroups and direct products.*

Proof of Theorem B. Firstly, from Lemma 3.1, \mathfrak{H}_σ is a hereditary formation.

Now we prove that $G \in \mathfrak{H}_\sigma$ if and only if G is soluble and the automorphism group induced by G on every its chief factor of order divisible by p is either a σ_i -group, where $p \notin \sigma_i$, or a $(\sigma_i \cup \sigma_j)$ -group for some different σ_i and σ_j such that $p \in \sigma_i$.

Necessity. Assume that this is false and let G be a counterexample of minimal order. Then G has a chief factor H/K of order divisible by p such that $A = G/C_G(H/K)$ is neither a σ_i -group, where $p \notin \sigma_i$, nor a $(\sigma_i \cup \sigma_j)$ -group, where $\sigma_i \neq \sigma_j$ and $p \in \sigma_i$. Since

$$G/C_G(H/K) \simeq (G/K)/(C_G(H/K)/K) = (G/K)/C_{G/K}(H/K)$$

and the hypothesis holds for G/K by Lemma 3.1, the choice of G implies that $K = 1$.

First we show that $H \neq C_G(H)$. Indeed, assume that $H = C_G(H)$. By hypothesis, every complete Hall σ -set $\mathcal{W} = \{W_1, \dots, W_t\}$ of G forms a σ -basis of G . Without loss of generality, we can assume that $p \in \pi(W_1)$. It is clear that $t > 2$. Since $H = C_G(H)$, H is the unique minimal normal subgroup of G and $H \not\leq \Phi(G)$ by [11, Ch.A, 9.3(c)] since G is soluble. Hence $H = O_p(G) = F(G)$ by [11, Ch.A, 15.6]. Then for some maximal subgroup M of G we have $G = H \rtimes M$. Let $V = W_3$. We now show that $V^x \leq C_G(W_2)$ for all $x \in G$. First note that $W_2V^x = V^xW_2$ is a Hall $(\sigma_2 \cup \sigma_3)$ -subgroup of G . Since $|G : M|$ is a power of p , any Hall σ_0 -subgroup of M , where $p \notin \pi_0$, is a Hall π_0 -subgroup of G . Hence we can assume without loss of generality that $W_2V^x \leq M$ since G is soluble. By hypothesis, $W_2(V^x)^y = (V^x)^yW_2$ for all $y \in G$, so

$$D = \langle (W_2)^{V^x} \rangle \cap \langle (V^x)^{W_2} \rangle$$

is subnormal in G by [14, 1.1.9(2)]. But $D \leq \langle W_2, V^x \rangle \leq M$, so

$$D^G = D^{HM} = D^M \leq M_G = 1$$

by [11, Ch. A, 14.3], which implies that $[W_2, V^x] = 1$. Thus $V^x \leq C_G(W_2)$ for all $x \in G$. It follows that $H \leq (W_3)^G \leq N_G(W_2)$ and therefore $W_2 \leq C_G(H) = H$, a contradiction. Hence $H \neq C_G(H)$.

Finally, let $D = G \times G$, $A^* = \{(g, g) | g \in G\}$, $C = \{(c, c) | c \in C_G(H)\}$ and $R = \{(h, 1) | h \in H\}$. Then $C \leq C_D(R)$, R is a minimal normal subgroup of A^*R and the factors $R/1$ and RC/C are (A^*R) -isomorphic. Moreover,

$$C_{A^*R}(R) = R(C_{A^*R}(R) \cap A^*) = RC,$$

so

$$A^*R/C = (RC/C) \rtimes (A^*/C),$$

where $A^*/C \simeq A$ and RC/C a minimal normal subgroup of A^*R/C such that $C_{A^*R/C}(RC/C) = RC/C$. As $H < C_G(H)$, we see that $|A^*R/C| < |G|$. On the other hand, by Lemma 3.1, the hypothesis holds for A^*R/C , so the choice of G implies that $A \simeq A^*/C$ is either a σ_i -group, where $p \notin \sigma_i$, or a $(\sigma_i \cup \sigma_j)$ -group for some different σ_i and σ_j such that $p \in \sigma_i$. This contradiction completes the proof of the necessity.

Sufficiency. Assume that this is false and let G be a counterexample of minimal order. Then G has a complete Hall set $\mathcal{W} = \{W_1, \dots, W_t\}$ of type σ such that for some i and j we have $W_i W_j \neq W_j W_i$. Let R be a minimal normal subgroup of G . Then:

- (1) $G/R \in \mathfrak{H}_\sigma$, so R is a unique minimal normal subgroup of G .

It is clear that the hypothesis holds for G/R , so $G/R \in \mathfrak{H}_\sigma$ by the choice of G . If G has a minimal normal subgroup $L \neq R$, then we also have $G/L \in \mathfrak{H}_\sigma$. Hence G is isomorphic to some subgroup of $(G/R) \times (G/L)$ by [6, I, 9.7]. It follows from Lemma 3.1 that $G \in \mathfrak{H}_\sigma$. This contradiction shows that we have Claim (1).

- (2) *The hypothesis holds for any subgroup E of G .*

Let H/K be any chief factor of G of order divisible by p such that $H \cap E \neq K \cap E$. Then $G/C_G(H/K)$ is either a σ_i -group, where $p \notin \sigma_i$, or a $(\sigma_i \cup \sigma_j)$ -group for some different σ_i and σ_j such that $p \in \sigma_i$. Let H_1/K_1 be a chief factor of E such that $K \cap E \leq K_1 < H_1 \leq H \cap E$. Then H_1/K_1 is a p -group and

$$EC_G(H/K)/C_G(H/K) \simeq E/(E \cap C_G(H/K))$$

is either a σ_i -group or a $(\sigma_i \cup \sigma_j)$ -group. Since

$$C_G(H/K) \cap E \leq C_E(H \cap E/K \cap E) \leq C_E(H_1/K_1),$$

$E/C_E(H_1/K_1)$ is also either a σ_i -group or a $(\sigma_i \cup \sigma_j)$ -group. Therefore the hypothesis holds for every factor H_1/K_1 of some chief series of E . Now applying the Jordan-Hölder Theorem for chief series we get Claim (2).

- (3) *R is a Sylow p -subgroup of G .*

Since $G/R \in \mathfrak{H}_\sigma$ by Claim (1),

$$(W_i R/R)(W_j R/R) = (W_j R/R)(W_i R/R),$$

so $W_i W_j R$ is a subgroup of G . Assume that R is not a Sylow p -subgroup of G and let $B = W_i W_j R$. Then $B \neq G$. On the other hand, the hypothesis holds for B by Claim (2). The choice of G implies that $B \in \mathfrak{H}_\sigma$, so $W_i W_j = W_j W_i$, a contradiction. Hence Claim (3) holds.

Final contradiction for sufficiency. In view of Claims (1) and (3), there is a maximal subgroup M of G such that $G = R \rtimes M$ and $M_G = 1$. Hence $R = C_G(R) = O_p(G)$ by [11, Ch.A, 15.6]. Since p does not divide $|G : R| = |G : C_G(R)|$ by Claim (3), the hypothesis implies that $M \simeq G/R$ is a Hall σ_k -group for some $\sigma_k \in \sigma$, so one of the subgroups W_i or W_j coincides with R . Thus $G = W_i W_j = W_j W_i$. This contradiction completes the proof of the sufficiency.

Finally we prove that \mathfrak{H}_σ is saturated if and only if $|\sigma| \leq 2$. It is clear that \mathfrak{H}_σ is a saturated formation for any σ with $|\sigma| \leq 2$. Now we show that for any σ such that $|\sigma| > 2$, the formation \mathfrak{H}_σ is not saturated.

Indeed, since $|\sigma| > 2$, there are primes $p < q < r$ such that for some distinct σ_i, σ_j and σ_k in σ we have $p \in \sigma_i, q \in \sigma_j$ and $r \in \sigma_k$. Let C_q and C_r be groups of order q and r , respectively. Let P_1 be a simple $\mathbb{F}_p C_q$ -module which is faithful for C_q , P_2 be a simple $\mathbb{F}_p C_r$ -module which is faithful for C_r . Let $H = P_1 \rtimes C_q$ and Q be a simple $\mathbb{F}_q H$ -module which is faithful for H . Let $E = (Q \rtimes H) \times (P_2 \rtimes C_r)$.

Let $A = A_p(E)$ be the p -Frattini module of E ([11, p.853]), and let G be a non-splitting extension of A by E . In this case, $A \subseteq \Phi(G)$ and $G/A \simeq E$. Then $G/\Phi(G) \in \mathfrak{H}_\sigma$, where $\sigma = \{\sigma_i, \sigma_j, \sigma_k\}$. By Corollary 1 in [15], $QP_1 P_2 = O_{p',p}(E) = C_E(A/\text{Rad}(A))$. Hence for some normal subgroup N of G we have $A/N \leq \Phi(G/N)$ and $G/C_G(A/N) \simeq C_q \times C_r$ is a $(\sigma_i \cup \sigma_j)$ -group. But neither $p \notin \sigma_i$ nor $p \in \sigma_j$. Hence $G \notin \mathfrak{H}_\sigma$ by the necessity. The theorem is proved.

References

- [1] A.N. Skiba, On σ -subnormal and σ -permutable subgroups of finite groups, *J. Algebra*, **436** (2015), 1–16.
- [2] A.N. Skiba, On some results in the theory of finite partially soluble groups, *Commun. Math. Stat.*, **4**(3) (2016), 281–309.
- [3] W. Guo, A.N. Skiba, On II-quasinormal subgroups of finite groups, *Monatsh. Math.*, DOI: 10.1007/s00605-016-1007-9.
- [4] A.N. Skiba, A generalization of a Hall theorem, *J. Algebra and its Application*, **15**(5) (2016), DOI: 10.1142/S0219498816500857.
- [5] Huppert B.: Zur Sylowstruktur Auflösbarer Gruppen, II. *Arch. Math.*, **15**, 251–257 (1964).

- [6] Huppert B.: Endliche Gruppen I, Springer-Verlag, Berlin-Heidelberg-New York, 1967.
- [7] Huppert B.: Zur Sylowstruktur Auflösbarer Gruppen, *Arch. Math.*, **12**, 161–169 (1961).
- [8] Asaad M., Heliel A. A.: On permutable subgroups of finite groups, *Arch. Math.*, **80**, 113–118 (2003).
- [9] Huppert B., Blackburn N.: Finite Groups III, Springer-Verlag, Berlin–New York, 1982.
- [10] Tyutyanov V. N.: On the Hall conjecture (Russian. English, Ukrainian summary) *Ukrain. Mat. Zh.* **54**(7), 981–990 (2002); translation in *Ukrainian Math. J.*, **54**(7), 1181–1191 (2002).
- [11] Doerk K., Hawkes T.: Finite Soluble Groups, Walter de Gruyter, Berlin, New York, 1992.
- [12] B.N. Knyagina, V.S. Monakhov. On π' -properties of finite groups having a Hall π -subgroup, *Siberian Math. J.*, **522** (2011), 398–309.
- [13] Gorenstein D.: Finite Groups, Harper & Row Publishers, New York-Evanston-London, 1968.
- [14] A. Ballester-Bolinches, R. Esteban-Romero, M. Asaad, *Products of Finite Groups*, Walter de Gruyter, Berlin-New York, 2010.
- [15] R. Griess, P. Schmid, The Frattini module, *Arch. Math.*, **30**, 256–266 (1978).